

INTEGRATION OF INTERVAL FUNCTIONS

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By

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INTRODUCTION

Many of the classical integrals of real point functions of one variable may be reduced to a single limiting process applied to corresponding multiple valued functions defined on linear intervals. The most important of these are the Riemann, Stieltjes, and Hellinger integrals. The interval functions to which they give rise may be written briefly as $f(\xi)\Delta x$, $f(\xi)\Delta g(x)$, and $\Delta g_1(x)\Delta g_2(x)/\Delta g_3(x)$ respectively. The limiting process in each case involves finite subdivisions for which the length of the greatest sub-interval goes to zero. Various modifications of these integrals have been proposed either by using a different limiting process or by considering a set of characteristic properties of the integral itself. The process which will be referred to as the sigma limit is more general than the norm limit mentioned above. Another variation is to allow the use of subdivisions with a denumerably infinite number of sub-intervals.

The first four sections of this paper are devoted to the application of the norm and sigma limits to interval functions summed over finite subdivisions. When these limits exist they give rise to two interval functions which have the properties of integrals. We designate them as the norm and sigma integrals respectively of $f(I)$. It is shown that for a certain class of functions which are called pseudo-additive the two integrals are equivalent and that if the norm integral exists the function belongs to this class. Another result is that if a function is integrable in the sigma sense the sigma integral of its absolute value also exists. This is not true in the case of the norm integral. A further property enjoyed by the sigma integral alone is that its existence on two adjacent intervals implies its existence on their sum.

Integrals of point functions which apparently require a second limiting process may be typified by that of Lebesgue. As originally defined this second process enters in the form of the measure of certain point sets. We make use of the Osgood idea of limits of uniformly bounded sequences of functions integrable in the sigma sense. In order to obtain a limit which gives the desired results it is found necessary to restrict our functions to

those which have values assigned to the various points of the interval. No loss in generality is suffered as long as the theory is concerned with the ultimate application to the integration of point functions. The fifth section develops the fundamental properties of sequences of interval functions.

In the last section two extended integrals are defined. The more general of these is an integral of the Stieltjes type. In particular it removes the restriction that $f(x)$ and $g(x)$ have no common point of discontinuity if the interval function $f(\xi)\Delta g(x)$ is to have an integral. The characteristic theorems dealing with the Lebesgue integral do not apply unless the interval function in question has the property of absolute continuity. This is made the basis of the definition of the second integral. An alternate theory which depends upon measure relative to additive interval functions is outlined. It has the advantage of presenting a single process of integration without any loss of generality so long as unbounded functions are left out of consideration.

Let me take this opportunity to thank Professor T. H. Hildebrandt for his patient supervision of the work. The subject of this investigation was also suggested by him. Dr. R. S. Burington of Case School of Applied Science read the manuscript and offered helpful suggestions.

CHAPTER I

PRELIMINARY EXPLANATIONS

1.1) Interval Functions.

Let y be the set of all closed intervals contained in the given closed linear interval $X = (a, b)$. An interval function, $f(I)$, is any relation which assigns one or more real numbers to each interval, I , of the range y . The set of numbers assigned to any particular interval will be assumed to be bounded, but the set of values of the function on the whole range is not necessarily so. $N(I) = (\text{length of } I)$ is an example of an interval function which is defined and single valued for all the intervals of any given range. Point functions furnish a wide variety of associated interval functions such as $\Delta x = x^n - x'$; $\Delta \varphi(x) = \varphi(x^n) - \varphi(x')$; $\varphi(\xi)\Delta g(x)$, $x' \leq \xi \leq x^n$; $\int_{x'}^{x^n} \varphi(x)dx$; $\sqrt{x'} \varphi(x)$; and $\varphi(\xi)\Delta g_1\Delta g_2/\Delta g_3$, together with their absolute values, and with proper restrictions on $\varphi(x)$ and $g(x)$ to satisfy the conditions of boundedness. In the above x' and x^n are the end points of the interval which serves as argument. $(\varphi(x') + \varphi(x^n))\Delta g(x)/2$ and the determinant

$$\begin{vmatrix} \varphi(\xi) & g(\xi) & 1 \\ \varphi(x') & g(x') & 1 \\ \varphi(x^n) & g(x^n) & 1 \end{vmatrix}$$

are further examples. We shall also have occasion to define and make use of certain interval functions associated with any given interval function. Consider, for instance, $M(f, I)$, or more simply $M(I)$ in the case of a particular function, the LUB¹ of the values of $f(I)$ on I . There are others analogous to the variation, oscillation, and upper and lower integrals of point functions.

1.2) Additive Functions.

Two intervals will be called adjacent, overlapping, or disjoint, according as they have one, many, or no points in common. Non-overlapping intervals are either adjacent or disjoint. The sum, $I' + I''$, of two adjacent

1. LUB and GLB stand for least upper bound and greatest lower bound, respectively.

intervals is the interval which contains all the points of each. We shall say that $f(I)$ is additive whenever the relation $f(I') + f(I'') = f(I' + I'')$ holds for every pair of adjacent intervals I' and I'' . In the case of multiple valued functions the equality sign will be understood to mean that for any two determinations of the function in two of the terms there is a corresponding value of the third term which satisfies the equation, and conversely. The first, second, fourth, and fifth of the examples on the preceding page are single valued additive functions. If one of two such functions is less than or equal to the other on every interval then the function which takes on all values between, inclusively, is multiple valued and additive.

1.3) Pseudo-additive Functions.

If $\lim |f(I') + f(I'') - f(I)| = 0$ as $N(I)$ goes to zero for all intervals I which contain the point x as an inner point we shall say that $f(I)$ is pseudo-additive at x . I' and I'' are the two adjacent sub-intervals of I whose common point is x . An interval function is pseudo-additive on the interval X if it has the property at every inner point. It is obvious that single valued additive functions are pseudo-additive. If $\varphi(x)$ and $g(x)$ are bounded the interval function $\varphi(\xi)\Delta g(x)$, $x' \leq \xi \leq x''$, is pseudo-additive at every point for which either of the two point functions is continuous. Any interval function which goes to zero with $N(I)$; i.e., any continuous interval function, has the property in question. The Riemann function $f(I) = \varphi(x)\Delta x$ is a special case of each of the above provided that the point function $\varphi(x)$ is bounded.

1.4) Subdivisions.

By a subdivision, σ , of the interval $X = (a, b)$ is meant an ordered set of $n + 1$ distinct points, $a = x_0 < x_1 < x_2 \dots < x_n = b$. For the sake of convenience we shall also refer to the set of n sub-intervals into which X is divided as the subdivision.

The product, $\sigma_1 \cdot \sigma_2$, of two subdivisions is the subdivision which contains all the points of σ_1 and σ_2 , and no others.

Two subdivisions are equal if and only if they consist of the same set of points.

The relation $\sigma \geq \sigma_0$ means that every point σ_0 is also a point of σ . This relation has the properties of transitivity and composition. That is, if $\sigma_3 \geq \sigma_2$,

and $\sigma_2 \geq \sigma_1$, then $\sigma_3 \geq \sigma_1$, and for any two subdivisions σ_1 and σ_2 there exists a third, σ_3 , such that $\sigma_3 \geq \sigma_1$, and $\sigma_3 \geq \sigma_2$.

The sum, $\sigma_3 + \sigma_2$, takes the place of the product when the subdivided intervals are non-overlapping. In case they are disjoint the intermediate interval is not to be taken as belonging to the sum. Strictly speaking, then, the sum is not always a subdivision. However, it will be found useful to refer to the sum of a set of subdivisions of a finite number of non-overlapping intervals as σ .

The norm of a subdivision, N_σ , is the length of the longest sub-interval.

1.5) Functions of Subdivisions.

A function of subdivisions is a relation which assigns numbers to subdivisions. We are interested here only in those of the form $F(\sigma) = \sum_\sigma f(I)$, where the subscript means that the interval function is summed over the sub-intervals of σ . Not all functions of subdivisions can be obtained in this way from interval functions because the values assumed by an interval function are fixed for any particular sub-interval regardless of the subdivision in which it occurs.

Two properties of interest are those of additivity and monotonicity. $F(\sigma)$ will be said to be additive if the relation $\sigma = \sigma_1 + \sigma_2$ implies that $F(\sigma) = F(\sigma_1) + F(\sigma_2)$. Again we are dealing with multiple valued functions and the same interpretation as before is to be given to the equality sign. $F(\sigma) = \sum_\sigma f(I)$ is necessarily additive by definition. A function of subdivisions is monotonic, non-increasing, or non-decreasing, if the relation $\sigma \geq \sigma_1$ implies either $F(\sigma) \leq F(\sigma_1)$ or $F(\sigma) \geq F(\sigma_1)$, respectively. With multiple valued functions the sign $\geq$ means that no value on the left is less than any value on the right.

1.6) Monotonic Interval Functions.

The interval function $f(I)$ will be said to be monotonic whenever the function $F(\sigma) = \sum_\sigma f(I)$ is monotonic. A necessary and sufficient condition that $f(I)$ have this property is that the same one of the two relations $f(I') + f(I'') \geq f(I)$ and $f(I') + f(I'') \leq f(I)$ hold on every pair of adjacent intervals I' and I'' . The interval I is their sum. If $f(I)$ is any interval

function then the associated function $L(f, I)$ or $L(I)$ $= \text{LUB } \Sigma_{\sigma} f(I)$ on all σ of I is non-increasing. The function $f(I) = |\Delta\phi(x)|$ is non-decreasing.¹ The Darboux functions $M\Delta x$ and $m\Delta x$ are useful examples of the two types of monotonic interval functions. They are also pseudo-additive when the point function from which they are derived is bounded.

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1. Monotonic interval functions are called upper and lower semi-additive, (halbadditiv), by Kolmogoroff, p. 692. Saks treats only single valued additive interval functions which can be derived from point functions of bounded variation by the relation $f(I) = \Delta\phi(x)$. He classifies them according to the properties of $\phi(x)$. He therefore designates non-negative and non-positive interval functions as monotonic non-decreasing and non-increasing respectively.

CHAPTER II

LIMITS OF $F(\sigma)$ 2.1) The Sigma Limit.¹

$\lim F(\sigma) = A$ is equivalent to the statement that for every positive ϵ there exists a σ_ϵ dependent upon ϵ such that for every sigma which satisfies the relation $\sigma \geq \sigma_\epsilon$ it follows that $|F(\sigma) - A| \leq \epsilon$. This definition is immediately extensible to $A = +\infty$ or $A = -\infty$ by the substitution of $F(\sigma) \geq \epsilon$ or $F(\sigma) \leq -\epsilon$ respectively for the inequality involving A . If $F(\sigma)$ is multiple valued the inequality must hold for all determinations of that function.

2.2) The Norm Limit.

$\lim_N F(\sigma) = A$ is equivalent to the statement that for every positive ϵ there exists a number N_ϵ greater than zero and dependent upon ϵ such that $|F(\sigma) - A| \leq \epsilon$ whenever $N_\sigma \leq N_\epsilon$. The extension to infinite limits is similar to the previous one.

2.3) Upper and Lower Limits.

The upper sigma limit, $\overline{\lim} F(\sigma) = A$, is defined by the two statements,

- 1) For every positive ϵ there exists a σ_ϵ such that $F(\sigma) \leq A + \epsilon$ whenever $\sigma \geq \sigma_\epsilon$.
- 2) For every positive ϵ and σ chosen independently there exists a $\sigma_0 \geq \sigma$, dependent upon ϵ and σ , such that $F(\sigma) \geq A - \epsilon$. The other three, $\overline{\lim} F(\sigma)$, $\lim F(\sigma)$, and $\lim_N F(\sigma)$ are analogous, and may have infinite values.^N It is to be noticed that all four exist, finite or infinite, for any given interval function.

2.4) Existence Properties.

The properties in this paragraph and the next are well known. When the mode of convergence is not specified the result holds with either type of limit. The

1. Moore and Smith.

existence of a limit means the existence of a finite limit unless the contrary is stated.

- 1) $\lim F(\sigma)$ is unique if it exists, finite or infinite.
- 2) The Cauchy condition, (CC)¹, is necessary and sufficient for the existence of $\lim F(\sigma)$.
- 3) A necessary and sufficient condition for the existence of $\lim F(\sigma)$, finite or infinite, is $\lim F(\sigma) = \underline{\lim} F(\sigma)$.
- 4) $\overline{\lim}_N F(\sigma) \geq \underline{\lim}_\sigma F(\sigma) \geq \underline{\lim}_\sigma F(\sigma) \geq \underline{\lim}_N F(\sigma)$.
- 5) If $\lim_N F(\sigma)$ exists, finite or infinite, then $\lim F(\sigma)$ exists and is equal to it.
- 6) If $F(\sigma)$ is single valued and monotonic then $\lim F(\sigma)$ always exists, finite or infinite, and is equal to the LUB or GLB of $F(\sigma)$ for all subdivisions of X .

2.5) Elementary Properties.

- 1) If c is a constant then $\lim cF(\sigma) = cA$, where $A = \lim F(\sigma)$.
- 2) If $\lim F_1(\sigma) = A_1$ and $\lim F_2(\sigma) = A_2$ then $\lim (F_1(\sigma) + F_2(\sigma)) = A_1 + A_2$, unless $A_1 = -A_2 = \infty$.
- 3) If $F(\sigma) \leq c$ for every σ then $\lim F(\sigma) \leq c$ if it exists, finite or infinite.
- 4) If $F_1(\sigma) \leq F_2(\sigma)$ for every σ then $\lim F_1(\sigma) \leq \lim F_2(\sigma)$ whenever the limits exist, finite or infinite.

1. (CC) for sigma limit. For every positive ϵ there exists a σ_ϵ such that $|F(\sigma') - F(\sigma'')| \leq \epsilon$ whenever $\sigma' \geq \sigma_\epsilon$ and $\sigma'' \geq \sigma_\epsilon$. There is an analogous statement for the condition relative to the norm limit.

CHAPTER III

INTEGRALS OF $f(I)$

3.1) The Sigma Integral.

If $\lim_{\sigma} \Sigma_{\sigma} f(I)$ exists, finite or infinite, on subdivisions of the closed interval X we call it the sigma integral of $f(I)$ on X , or $\sigma \int_X f(I)$. For a given $f(I)$ it is seen that $\sigma \int_X f(I)$ is a single valued interval function on the range of intervals for which it is defined. The upper and lower sigma integrals, $\sigma \overline{\int}_X f(I)$ and $\sigma \underline{\int}_X f(I)$ respectively, are the corresponding upper and lower limits taken in the sigma sense.

3.2) The Norm Integral.

$N \int_X f(I)$, the norm integral of $f(I)$ on X , as well as the corresponding upper and lower norm integrals have like definitions in terms of norm limits. When we say that the integral of $f(I)$ exists on an interval its value will be understood to be finite. If no mode of convergence is specified the statement involving the integral will hold for either.

3.3) Elementary Theorems.

The following are immediately evident from the contents of paragraphs 2.4) and 2.5).

Theorem 3.31. $\int_X f(I)$ is unique if it exists, finite or infinite.

Theorem 3.32. The (CC) is necessary and sufficient for the existence of $\int_X f(I)$.

Theorem 3.33. For any given interval function the two pairs of upper and lower integrals always exist, finite or infinite.

Theorem 3.34. A necessary and sufficient condition for the existence of $\int_X f(I)$, finite or infinite, is $\overline{\int}_X f(I) = \underline{\int}_X f(I)$.

Theorem 3.35. $N \int_X f(I) \geq \sigma \overline{\int}_X f(I) \geq \sigma \underline{\int}_X f(I) \geq N \int_X f(I)$.

Theorem 3.36. If $N \int_X f(I)$ exists, finite or infinite, then $\sigma \int_X f(I)$ exists and is equal to it.

Theorem 3.37. If $f(I)$ is single valued and monotonic then $\sigma \int_X f(I)$ exists, finite or infinite, and is equal to the LUB or GLB of $\Sigma_{\sigma} f(I)$ on all subdivisions of X .

Theorem 3.38. If c is a constant and $\int_X f(I)$ exists, finite or infinite, then $\int_X cf(I) = c \int_X f(I)$.

Theorem 3.39. If $\int_X f_1(I)$ and $\int_X f_2(I)$ exist, finite or infinite, and $f(I) = f_1(I) + f_2(I)$ then $\int_X f(I) = \int_X f_1(I) + \int_X f_2(I)$. The two given integrals may not be infinite and of opposite sign.

Theorem 3.310. If $f_1(I) \leq f_2(I)$ for every I then $\int_X f_1(I) \leq \int_X f_2(I)$ whenever the two exist, finite or infinite.

CHAPTER IV

PROPERTIES OF INTEGRALS

4.1) Upper and Lower Associated Functions.

Let $M(f, I)$, or $M(I)$, be the LUB of the values of $f(I)$ on I , and $m(f, I)$, or $m(I)$, the GLB. These two functions are single valued and are additive when $f(I)$ is additive.

Theorem 4.11. A necessary and sufficient condition that $\int_X f(I)$ exist, finite or infinite, is that $\int_X M(I)$ and $\int_X m(I)$ exist, finite or infinite, and be equal.

Necessity. Choose σ_e such that $|\Sigma_\sigma f(I) - \sigma \int_X f(I)|$ is less than $e/2$ whenever the relation $\sigma \geq \sigma_e$ holds. There exists a choice of the values of $f(I)$, call it $f'(I)$, such that $(\Sigma_\sigma M(I) - \Sigma_\sigma f'(I)) \leq e/2$. Since the first inequality is true for all determinations of $\Sigma_\sigma f(I)$ we have $|\Sigma_\sigma M(I) - \sigma \int_X f(I)| \leq e$. Therefore $\sigma \int_X f(I) = \sigma \int_X M(I)$. In the same way it can be shown that $\sigma \int_X f(I) = \sigma \int_X m(I)$. The modifications necessary for application to the N integral and to the infinite cases are slight.

Sufficiency. For every σ we have the inequality $\Sigma_\sigma M(I) \geq \Sigma_\sigma f(I) \geq \Sigma_\sigma m(I)$. It follows that $\int_X M(I) \geq \int_X f(I) \geq \int_X m(I)$. Since the extremes are equal the integral exists, finite or infinite, by Theorem 3.34.

4.2) The Variation Functions.

Let $V(f, I)$ and $v(f, I)$ be the LUB and GLB, respectively, of $\Sigma_\sigma f(I)$ for all subdivisions of I and for all values of the function on the separate sub-intervals. It is noticed that the one is non-increasing and the other non-decreasing. In order to make them satisfy the conditions of boundedness it may be necessary to restrict their ranges. If $\Sigma_{\sigma_0} V(f, I)$ is finite for any particular subdivision of X then $V(f, I)$ is defined and finite on the range y_0 consisting of all sub-intervals of the subdivisions $\sigma \geq \sigma_0$. If no such σ_0 exists the function is undefined. Similar statements obtain for

$v(f, I)$. For a particular function we may write $V(I)$ and $v(I)$ for brevity.

Lemma. A necessary and sufficient condition that $V(f, I)$ be defined on X is that $\sigma \int_X f(I)$ be finite.

Necessity. Let $\sigma \int_X f(I) = M/2 < \infty$. By definition there exists a σ_0 such that $\Sigma_{\sigma} f(I) \leq M$, whenever $\sigma \geq \sigma_0$. Therefore $\Sigma_{\sigma_0} V(I) \leq M$.

Sufficiency. $\Sigma_{\sigma} f(I) \leq \Sigma_{\sigma_0} V(I) = M < \infty$. Here σ is any subdivision such that $\sigma \geq \sigma_0$. But there exists a $\sigma_1 \geq \sigma_0$ such that $\sigma \int_X f(I) \leq \Sigma_{\sigma_1} f(I) + M$. Therefore $\sigma \int_X f(I) \leq 2M$.

Theorem 4.21. If $V(f, I)$ is defined on X then $\sigma \int_X f(I) = \sigma \int_X V(f, I)$.

Proof. It will first be shown that the inequality $\sigma \int_X f(I) \leq \sigma \int_X V(I)$ holds. Choose any positive e . There exists a subdivision σ_0 such that

$$\Sigma_{\sigma_0} V(I) \leq \sigma \int_X V(I) + e/2.$$

There can be found a $\sigma_1 \geq \sigma_0$ for which

$$\sigma \int_X f(I) \leq \Sigma_{\sigma_1} f(I) + e/2.$$

Since $\Sigma_{\sigma_1} f(I) \leq \Sigma_{\sigma_0} V(I)$ it follows that $\sigma \int_X f(I)$ is not less than $\sigma \int_X V(I) + e$ for any positive e .

In order to reverse the above inequality choose a σ_0 such that

$$\Sigma_{\sigma} f(I) \leq \sigma \int_X f(I) + e/2$$

whenever $\sigma \geq \sigma_0$. There exists a $\sigma_1 \geq \sigma_0$ and a set of values $f'(I)$ of $f(I)$ on σ_1 for which

$$\Sigma_{\sigma_0} V(I) \leq \Sigma_{\sigma_1} f'(I) + e/2.$$

From these two inequalities and the monotonicity of $V(I)$ we conclude that $\sigma \int_X V(I) \leq \sigma \int_X f(I) + e$. Since the integrals are constant and the inequality holds in both directions the theorem is established. In the same way it can be shown that $\sigma \int_X f(I) = \sigma \int_X v(f, I)$.

Corollary. A necessary and sufficient condition for the existence of $\sigma \int_X f(I)$, finite or infinite, is that $\sigma \int_X V(f, I) = \sigma \int_X v(f, I)$. When $V(I)$ or $v(I)$ is undefined we define its sigma integral to be $+\infty$ or $-\infty$, respectively.

Proof. This corollary is in immediate consequence of Theorem 3.34.

4.3) The Oscillation Function.

Let $\omega(f, I)$ or $\omega(I)$ be the LUB of $|\Sigma_{\sigma'} f(I) - \Sigma_{\sigma''} f(I)|$ for all pairs of subdivisions σ' and σ'' of I . Then if $V(I)$ and $v(I)$ are defined on X it follows that on their common range $\omega(I) = V(I) - v(I) \geq 0$.

Theorem 4.31. A necessary and sufficient condition for the existence of $\int_X f(I)$ is $\int_X \omega(f, I) = 0$.

Proof. The truth of the theorem for the sigma integral is evident from the corollary to Theorem 4.21.

Necessity. If $N \int_X f(I)$ exists we have, applying the (CC),

$$|\Sigma_{\sigma'} f(I) - \Sigma_{\sigma''} f(I)| \leq e/2$$

for any two subdivisions, σ' and σ'' , whose norms are not greater than a certain N_e . Let σ be any subdivision such that $N_{\sigma} \leq N_e$. If n is the number of points of σ and I_i its i -th sub-interval there exists for every i a pair of subdivisions σ_i' and σ_i'' of I_i , for which

$$\omega(I_i) \leq \Sigma_{\sigma_i'} f(I) - \Sigma_{\sigma_i''} f(I) + e/2n.$$

Let $\sigma' = \Sigma \sigma_i'$ and $\sigma'' = \Sigma \sigma_i''$. Their norms are not greater than N_e . But

$$\begin{aligned} \Sigma_{\sigma} \omega(I) &\leq \Sigma (\Sigma_{\sigma_i'} f(I) - \Sigma_{\sigma_i''} f(I)) + e/2 \\ &= \Sigma_{\sigma'} f(I) - \Sigma_{\sigma''} f(I) + e/2 \leq e. \end{aligned}$$

Since $\omega(I)$ is never negative we have proved that its norm integral is zero.

Sufficiency. If $N \int_X \omega(I) = 0$ it follows that for any subdivision σ whose norm is not greater than a certain

N_e we have

$$\Sigma_{\sigma} \omega(I) \leq e/2.$$

Let σ' and σ'' be any two subdivisions which have this property. Then $N_{\sigma} \leq N_e$, where $\sigma = \sigma' \cdot \sigma''$. If σ_1' and σ_1'' are the two sets of subdivisions introduced into the i -th sub-interval of σ' and σ'' respectively in making up their product we obtain the following,

$$|\Sigma_{\sigma'} f(I) - \Sigma_{\sigma} f(I)| \leq \Sigma_{\sigma'} |f(I) - \Sigma_{\sigma_1'} f(I)| \leq \Sigma_{\sigma'} \omega(I)$$

$$|\Sigma_{\sigma''} f(I) - \Sigma_{\sigma} f(I)| \leq \Sigma_{\sigma''} |f(I) - \Sigma_{\sigma_1''} f(I)| \leq \Sigma_{\sigma''} \omega(I).$$

$$|\Sigma_{\sigma'} f(I) - \Sigma_{\sigma''} f(I)| \leq \Sigma_{\sigma'} \omega(I) + \Sigma_{\sigma''} \omega(I) \leq e.$$

The last inequality is the (CC) for the N integral.

4.4) Comparison of the Norm and Sigma Integrals.

Theorem 4.41. For the existence of $N \int_X f(I)$ it is necessary and sufficient that $\sigma \int_X f(I)$ exist and that $f(I)$ be pseudo-additive on X .

Necessity. The existence of $N \int_X f(I)$ implies that of $\sigma \int_X f(I)$. It must be shown that $f(I)$ is pseudo-additive on X . Assume x_1 to be an inner point of X at which $f(I)$ is not pseudo-additive. There exists a positive e_1 such that for every N an interval I_N can be found with the following properties, I_N contains x_1 as an inner point, $N(I_N) \leq N$, and if I' and I'' are the two sub-intervals into which I_N is divided by x_1 then

$$|f(I') + f(I'') - f(I)| > e_1.$$

Let σ be any subdivision X whose norm is less than N and which has the interval I_N as one of its sub-intervals. The addition of x_1 to σ alters the value of $\Sigma_{\sigma} f(I)$ by at least e_1 . Since N is arbitrary the (CC) for the norm integral cannot hold and the assumption that $f(I)$ is not pseudo-additive at x_1 contradicts the hypothesis that $N \int_X f(I)$ exists.

Sufficiency. Because $\sigma \int_X f(I)$ exists we have

$$\Sigma_{\sigma} f(I) - \sigma \int_X f(I) \leq e/2$$

whenever $\sigma' \geq \sigma_e$. Let n be the number of points of σ_e . Since $f(I)$ is pseudo-additive there exists an N_e such that if I contains a point of σ_e in its interior and if $N(I) \leq N_e$ then

$$|f(I') + f(I'') - f(I)| \leq e/2n.$$

Let σ be any subdivision of X whose norm is not greater than N_e . If $\sigma' = \sigma \cdot \sigma_e$ it follows that $\sigma' \geq \sigma_e$, and

$$|\Sigma_{\sigma'} f(I) - \Sigma_{\sigma} f(I)| \leq ne/2n = e/2.$$

This inequality together with the first is equivalent to

$$|\Sigma_{\sigma} f(I) - \sigma \int_X f(I)| \leq e.$$

Therefore $N \int_X f(I) = \sigma \int_X f(I)$ by definition.

In consequence of Theorem 4.31 we have the following

Corollary. If $\sigma \int_X f(I)$ exists a necessary and sufficient condition that $N \int_X f(I)$ exist is that $\omega(f, I)$ be pseudo-additive.

The Riemann sum, $\Sigma_{\sigma} f(x) \Delta x$, is pseudo-additive. So is the Stieltjes sum, $\Sigma_{\sigma} f(x) \Delta g(x)$, provided $f(x)$ and $g(x)$ are bounded and have no common points of discontinuity. Hence there is no distinction between the norm and sigma modes of convergence in these two cases. If $f(I) = (\varphi(x') + \varphi(x'')) \Delta g(x)/2$ the expression $(f(I') + f(I'') - f(I))$ reduces to the determinant

$$\begin{vmatrix} g(x) & \varphi(x) & 1 \\ g(x') & \varphi(x') & 1 \\ g(x'') & \varphi(x'') & 1 \end{vmatrix} \cdot \frac{1}{2}.$$

The vanishing of this expression at every point x interior to X as x' and x'' approach it independently from opposite sides is then a necessary and sufficient condition for the equivalence of the two integrals in this case.

1. $\int_a^b (1/2)(f(x') + f(x'')) dg(x)$ is called the mean integral by Kaltenborn.

4.5) Additivity.

Theorem 4.51. $\sigma \int_I f(I)$ is an additive interval function. Its existence on any pair of adjacent intervals implies its existence on their sum and conversely.

Proof. It is sufficient to show that $\sigma \int_I f(I)$ is additive. The rest follows because the lower integral is likewise additive and not less than upper integral on any interval.

Let I' and I'' be any pair of adjacent intervals with the sum I . It will first be shown that

$$\sigma \int_I f(I) \leq \sigma \int_{I'} f(I) + \sigma \int_{I''} f(I).$$

There exists a σ_e' of I' and a σ_e'' of I'' such that $\Sigma_{\sigma'} f(I) \leq \sigma \int_{I'} f(I) + e/3$ and $\Sigma_{\sigma''} f(I) \leq \sigma \int_{I''} f(I) + e/3$ for every pair of subdivisions $\sigma' \geq \sigma_e'$ and $\sigma'' \geq \sigma_e''$. But there exists a subdivision σ of I such that $\sigma \geq \sigma_e' + \sigma_e''$, and $\sigma \int_I f(I) \leq \Sigma_{\sigma} f(I) + e/3$. Let σ' and σ'' be the parts of σ on I' and I'' respectively. On combining the inequalities above we get

$$\sigma \int_I f(I) \leq \sigma \int_{I'} f(I) + \sigma \int_{I''} f(I) + e.$$

Since e is an arbitrary positive constant it may be dropped.

To reverse the inequality let σ_e be a subdivision of I such that

$$\sigma \int_I f(I) \geq \Sigma_{\sigma} f(I) - e/3$$

whenever $\sigma \geq \sigma_e$. There exist subdivisions σ' and σ'' of I' and I'' respectively such that $\sigma = \sigma' + \sigma'' \geq \sigma_e$, $\Sigma_{\sigma'} f(I) \geq \sigma \int_{I'} f(I) - e/3$, and $\Sigma_{\sigma''} f(I) \geq \sigma \int_{I''} f(I) - e/3$. The result of these three inequalities is

$$\sigma \int_I f(I) \geq \sigma \int_{I'} f(I) + \sigma \int_{I''} f(I) - e.$$

It is to be noticed that the N integral may exist on a pair of adjacent intervals without existing on their sum. This will occur only when $f(I)$ fails to be pseudo-additive at their common point. The following two theorems depend upon Theorems 4.41 and 4.51.

Theorem 4.52. If $N \int_X f(I)$ exists then $N \int_I f(I)$ exists on every sub-interval, I , of X and is an additive function of intervals.

Theorem 4.53. If $f(I)$ is integrable in the norm sense on a pair of adjacent intervals and pseudo-additive at their common point then the norm integral of $f(I)$ exists on their sum.

The point of view that the integral of a function must be single valued is not the only one possible. The totality of numbers which lie between the upper and lower integrals inclusively on each interval may be considered as the integral. Then the integral always exists and is additive when taken in the sigma sense. A single valued integral then corresponds to the integral of the former definition.¹

4.6) Absolute integrability.

A function $f(I)$ will be called essentially bounded, or bounded, on X if the variation functions $V(|f|, I)$ and $v(|f|, I)$ are defined there. This is equivalent to the statement that there exists a positive number M and a subdivision σ_0 of X such that $\Sigma_{\sigma} |f(I)| \leq M$ whenever $\sigma \geq \sigma_0$.

Theorem 4.61. If $\sigma \int_X f(I)$ exists and $f(I)$ is essentially bounded on X then $\sigma \int_X |f(I)|$ also exists.

Proof. Consider first the special case in which the given function is single valued. Let $P(I)$ and $Q(I)$ be the non-negative functions defined by the equations

$$P(I) \equiv (|f(I)| + f(I))/2$$

$$Q(I) \equiv (|f(I)| - f(I))/2.$$

Adding and subtracting in turn gives

$$f(I) \equiv P(I) - Q(I), \quad |f(I)| \equiv P(I) + Q(I).$$

If $A = \sigma \int_X f(I)$, which exists by hypothesis, then for every positive ϵ there exists a σ_ϵ such that

1. Kolmogoroff, p. 692.

$$\Sigma_{\sigma}P(I) - \Sigma_{\sigma}Q(I) \equiv \Sigma_{\sigma}f(I) \leq A + e/2$$

$$\Sigma_{\sigma}P(I) - \Sigma_{\sigma}Q(I) \equiv \Sigma_{\sigma}f(I) \geq A - e/2$$

whenever $\sigma \geq \sigma_e$. For convenience let the above inequalities be written in the form

$$1) \Sigma_{\sigma}Q(I) + A - e/2 \geq \Sigma_{\sigma}P(I) - e$$

$$2) \Sigma_{\sigma}P(I) \geq \Sigma_{\sigma}Q(I) + A - e/2.$$

Now let σ_1 and σ_2 be any two subdivisions which satisfy the relation $\sigma_2 \geq \sigma_1 \geq \sigma_e$. If σ_0 is the subdivision of X obtained by adding to σ_1 those points of σ_2 which fall within the sub-intervals of σ_1 on which $Q(I) = 0$ then $\Sigma_{\sigma_0}Q(I) \geq \Sigma_{\sigma_1}Q(I)$ because $Q(I)$ is non-negative. Its sum cannot be decreased by adding points to the sub-intervals on which it has the value zero. Since $P(I) = 0$ on the complementary intervals of σ_1 , which are the same for σ_0 , it is seen that σ_2 can be derived from σ_0 by adding points to the sub-intervals of σ_0 on which $P(I) = 0$. Therefore $\Sigma_{\sigma_2}P(I) \geq \Sigma_{\sigma_0}P(I)$. By setting $\sigma = \sigma_0$ in 2) and $\sigma = \sigma_1$ in 1) we obtain

$$3) \Sigma_{\sigma_2}P(I) \geq \Sigma_{\sigma_1}P(I) - e$$

for every pair of subdivisions σ_1 and σ_2 which satisfy the relation $\sigma_2 \geq \sigma_1 \geq \sigma_e$. By hypothesis $L = \sigma_X^f P(I) \leq \sigma_X^f |f(I)|$ is finite. There exists a $\sigma_1 \geq \sigma_e$ such that $\Sigma_{\sigma_1}P(I) \geq L - e$. On comparing this last inequality with 3) we find that for every $\sigma \geq \sigma_1$ it is true that $\Sigma_{\sigma}P(I) \geq L - 2e$. But this is precisely the condition that L , the upper integral of $P(I)$, be equal to the lower integral. Therefore $\sigma_X^f P(I)$ exists. Since $Q(I) = P(I) - f(I)$ it is also integrable in the sigma sense. $|f(I)| = P(I) + Q(I)$ has then a finite integral.

Now let $f(I)$ be any interval function which satisfies the hypotheses of the theorem. Its upper and lower functions, defined in paragraph 4.1) have equal integrals on X by Theorem 4.11. If $M'(I)$ and $m'(I)$ are the upper and lower functions, respectively, of $|f(I)|$ then for every interval I

$$0 \leq M'(I) - |M(I)| \leq M(I) - m(I).$$

Therefore

$$\begin{aligned} 0 \leq \sigma \int_X [M'(I) - M(I)] &\leq \sigma \bar{\int}_X [M'(I) - M(I)] \\ &\leq \sigma \bar{\int}_X [M(I) - m(I)] = 0. \end{aligned}$$

Since $\sigma \int_X |M(I)|$ exists we see from the above inequalities that $\sigma \int_X M'(I)$ exists and is equal to it. In the same way it can be shown that $\sigma \int_X m'(I) = \sigma \int_X |m(I)|$. Finally, from the inequality

$$M'(I) - m'(I) \leq M(I) - m(I)$$

it follows that

$$\sigma \int_X M'(I) = \sigma \int_X m'(I),$$

a sufficient condition for the existence of $\sigma \int_X |f(I)|$ by Theorem 4.11.

If $\varphi(x)$ is a function of bounded variation on X the difference function $f(I) = \varphi(x'') - \varphi(x')$ is single valued and additive. Therefore both norm and sigma integrals exist. The sigma integral of $|f(I)|$ exists by the theorem just proved. It represents the total variation of $\varphi(x)$ on X . If there exists an interior point, x_1 , of X such that $\varphi(x_1)$ is greater than the upper limit of $\varphi(x)$ as x approaches x_1 on either side then the absolute value of $f(I)$ is not a pseudo-additive function at x_1 and $N \int_X |f(I)|$ fails to exist. From this example it is seen that Theorem 4.61 does not hold in general for the norm integral.

It is known that if the Stieltjes integral $\sigma \int_X f(x) dg(x)$ exists then $\sigma \int_X |f(x)| dVg(x)$ does also when $g(x)$ is a function of bounded variation. If $g(x)$ is not of bounded variation the second integral has no meaning. Theorem 4.61, however, offers an extension in this case by asserting the existence of $\sigma \int_X |f(x) dg(x)|$, or $\sigma \int_X |f(x)| \cdot |dg(x)|$, equal to the integral involving the variation function when it is defined.

Corollary 1. $\left| \int_X f(I) \right| \leq \sigma \int_X |f(I)|$ when the latter is finite.

Proof. If the integral on the left is taken in the sigma sense the inequality follows from the fact that

$$|\Sigma_{\sigma} f(I)| \leq \Sigma_{\sigma} |f(I)|.$$

When the norm integral is understood the sigma integral is equal to it and the result still holds.

Corollary 2. Any function integrable in the sigma sense and essentially bounded can be expressed as the difference of two non-negative integrable functions.

Corollary 3. If $f(I)$ and $\varphi(I)$ are bounded and integrable in the sigma sense their logical sum and product are also.

Proof. When both functions are single valued their logical sum and product are equal respectively to one half of the expressions

$$f(I) + \varphi(I) + |f(I) - \varphi(I)| \quad \text{and}$$

$$f(I) + \varphi(I) - |f(I) - \varphi(I)|.$$

Since these are linear combinations of integrable functions they are integrable. When the given functions are multiple valued it may be advantageous to define the logical sum and product in such a way that only a subset of the above expressions is considered. The corollary holds as long as one value is left to each interval of their range. For example, let $f(I) = f(\xi)\Delta x$ and $\varphi(I) = \varphi(\xi)\Delta x$, where $f(x)$ and $\varphi(x)$ are Riemann integrable point functions. The Riemann integrals of the logical sum and product of these point functions exist in consequence of this corollary.

4.7) Absolutely Continuous Functions.

Up to this point the integral has been defined only on closed intervals, or what amounts to the same thing, on fundamental sets consisting of a finite number of non-overlapping closed intervals. In order to extend the definition to include all bounded sets measurable in the sense of Lebesgue we introduce absolute continuity.

An interval function $f(I)$ will be called absolutely continuous on X if $\lim \Sigma_J |f(I)| = 0$ on all fundamental sets J contained in X as $m(J)$, the sum of

the lengths of the intervals of J , goes to zero. A sequence $f_n(I)$ is equi-absolutely continuous if each function is absolutely continuous independently of n .

Theorem 4.71. If $f(I)$ is absolutely continuous its integral is also.

Proof. Since absolutely continuous functions are pseudo-additive there is no distinction between their norm and sigma integrals. Let $F(I) = \int_X |f(I)|$. For every positive ϵ there exists a σ_ϵ such that

$$|\Sigma_{\sigma} |f(I)| - \Sigma_{\sigma} F(I)| \leq \epsilon/2$$

whenever $\sigma \geq \sigma_\epsilon$. Also $\Sigma_J |f(I)| \leq \epsilon/2$ whenever $m(J) \leq m_\epsilon$. Let J' be the fundamental set obtained from J by adding to it all the points of σ_ϵ which belong to any of its intervals. Then

$$|\Sigma_{J'} |f(I)| - \Sigma_{J'} F(I)| \leq \epsilon/2, \text{ and } \Sigma_{J'} |f(I)| \leq \epsilon/2$$

because $m(J') = m(J) \leq m_\epsilon$. Because $F(I)$ is additive

$$\Sigma_J F(I) = \Sigma_{J'} F(I) \leq \Sigma_{J'} |f(I)| + \epsilon/2 \leq \epsilon.$$

Therefore

$$\Sigma_J \left| \int_X f(I) \right| \leq \Sigma_J F(I) \leq \epsilon.$$

It is to be noticed that the same m_ϵ may be used both for the function and its integral. For this reason the property of equi-absolute continuity carries over from a sequence of functions to the sequence of corresponding integrals.

If $f(I)$ is non-negative we define its integral on any open set O to be the LUB of the integrals on all fundamental sets contained in O . For a measurable set E let $\int_E f(I) = \text{GLB } \int_O f(I)$ on all open sets O which contain E . We are speaking only of the sigma integral here. There is no loss of generality in treating non-negative functions because of corollary 2 of Theorem 4.61. If measure is understood in the sense of Lebesgue the definition of the integral on sets of points is consistent only for absolutely continuous functions. In this

case it is seen that the properties of additivity and absolute continuity carry over to measurable sets.

Theorem 4.72. If E_n is a sequence of measurable sets with the limiting set E then $\lim \int_{E_n} f(I) = \int_E f(I)$.

Proof. Let $E_n' = E - EE_n$ and $E_n'' = E_n - EE_n$. Then $E = EE_n + E_n'$ and $E_n = EE_n + E_n''$. But

$$\begin{aligned} \left| \int_{E_n} f(I) - \int_E f(I) \right| &= \left| \int_{E_n''} f(I) - \int_{E_n'} f(I) \right| \\ &\leq \left| \int_{E_n''} f(I) \right| + \left| \int_{E_n'} f(I) \right|. \end{aligned}$$

Because of the absolute continuity of the integral the final terms vanish as n becomes infinite.

CHAPTER V

POINT INTERVAL FUNCTIONS

5.1) Definitions.

In the preceding sections closed intervals were the primary consideration. Points were brought in only where necessary, for the most part as the end points of intervals. The concept of integration can be extended considerably when the value of the function depends explicitly upon the points of the interval, as it does in the integration of point functions.

Let the point interval function $f(I, x)$ be an interval function with one and only one value assigned to each point x of the interval I . All the functions specifically mentioned heretofore are of this type. In particular, single valued functions are those which are independent of the points of the interval.

Interval functions represented by the letter "g", as $g(I)$, $g_0(I)$, and $g'(I)$; will be assumed to be non-negative, single valued, and their sums uniformly bounded on all subdivisions.

A nest is an infinite sequence of closed intervals, each one contained in the preceding, and such that there is but one point common to all of the intervals. This will be called the associated point.

A property holds almost everywhere if the set of points where it fails (in at least one interval) is of measure zero in the Lebesgue sense.

If $\varphi_n(I, x)$ is a given sequence, then every nest and $g(I)$ determines a pair of number sequences,

$$A_n = \varphi_n(I_n, y)/g(I_n) \text{ and } B_n = \underline{\varphi}_n(I_n)/g(I_n).$$

$\underline{\varphi}_n(I) = \text{GLB } \varphi_n(I, x)$ for all x of I , and y is the associated point of the nest. Where the numerator of the expression for A_n or B_n is zero, $g(I_n)$ may take the value zero. In such a case the quotient is defined to be zero also. If $f_n(I, x)$ is a sequence and there exist functions $f(I, x)$ and $g(I)$ such that $\lim A_n = 0$ for every nest, where A_n is formed for the function $\varphi_n(I, x) = |f_n(I, x) - f(I, x)|$, we shall write $\lim f_n(I, x) = f(I, x)$.

$\lim B f_n(I, x) = f(I, x)$ is equivalent to $\lim B_n = 0$. Since $B_n \leq A_n$ the second limit is implied by the first. The weaker of the two is not unique in general unless the limit function is integrable in the σ sense.

5.2) Sequences of Point Interval Functions.

All the sequences considered will be assumed to be uniformly bounded and the elements integrable σ . There is no loss of generality in restricting ourselves to non-negative functions, (see corollary 2 to Theorem 4.61).

Lemma. Let $[\sigma_n]$ be a monotonic sequence of subdivisions of X whose norm approaches zero as n increases. If $g(I)$ is such that there exist a positive β and a sub-set J_n of the intervals of σ_n for which $\sum_{J_n} g(I) > \beta$ with every n , then there exists a nest of intervals $[J_n]$.

Proof. It is sufficient to consider $g(I)$ non-increasing because $V(g, I) = \text{LUB } \sum_{\sigma} g(I)$ on all σ of I is not less than $g(I)$ on any interval, is non-increasing, and has the other properties required by the lemma.

Consider the closed interval X' of length $g(X)$. Corresponding to $I_1 \dots I_K$ of σ_1 , lay out in order on X' the closed intervals $I'_1 \dots I'_K$ with no overlapping, in each case taking $N(I') = g(I)$. This is possible because $g(I)$ is non-increasing. Repeat the process with the subdivisions of the sub-intervals of σ_1 imposed by the points of σ_2 , and continue with each σ_n in turn. Every nest of X' made up of these intervals has a corresponding nest on X . Each set of J_n gives rise to a set J'_n of intervals on X' of total length greater than β . The set of points common to an infinity of the sets J'_n has a lower measure not less than β .¹ Any one of these points determines a nest of intervals of $[J'_n]$. The corresponding nest on X is the one required.

Theorem 5.21. If $\lim B \varphi_n(I, x) = 0$, then $\lim \int_X \varphi_n(I, x) = 0$.

Proof. Because $g(I)$ may be increased on any interval without affecting the limit of B_n it may be assumed that $g(I) \geq \varphi_n(I, x)$ for every n . Suppose the conclusion of the theorem to be false. Then there exists a positive e_0 and a subsequence such that $\int_X \varphi_n(I, x) > e_0$ for every

1. Hobson, Vol. I, p. 175.

n. Also $\Sigma_{\sigma} \varphi_n(I, x) \geq \int_X \varphi_n(I, x) - e_0/2 > e_0/2$ whenever $\sigma \geq \sigma'_n$. Take $N_{\sigma'_n} \leq 1/n$. Then

$$1) \Sigma_{\sigma_n} \varphi_n(I, x) > e_0/2, \sigma_n = \sigma'_1 \dots \sigma'_n, N_{\sigma_n} \leq 1/n.$$

All the subdivisions involved must be taken so that the corresponding sums are finite. This can be done because the functions are uniformly bounded by hypothesis. Now separate the intervals of σ_n into the sets J_n and J'_n such that $\varphi_n(I) > e_0 g(I)/8M$ on J_n and $\varphi_n(I) \leq e_0 g(I)/8M$ on J'_n . M is the upper bound of $\Sigma_{\sigma_n} g(I)$, which is finite by hypothesis. Each interval contains a point x' such that $\varphi_n(I, x') \leq e_0 g(I)/8M + \varphi_n(I)$. On J'_n we have $\varphi_n(I, x') \leq e_0 g(I)/4M$. Therefore $\Sigma_{J'_n} \varphi_n(I, x') \leq \Sigma_{J'_n} e_0 g(I)/4M \leq e_0/4$. But

$$\begin{aligned} \Sigma_{\sigma_n} \varphi_n(I, x') &= \Sigma_{J_n} \varphi_n(I, x') + \Sigma_{J'_n} \varphi_n(I, x') \\ &\leq \Sigma_{J_n} \varphi_n(I, x') + e_0/4. \end{aligned}$$

Then by 1),

$$\Sigma_{J_n} \varphi_n(I, x') \geq \Sigma_{J_n} \varphi_n(I, x') - e_0/4 > e_0/4.$$

$$2) \Sigma_{J_n} g(I) \geq \Sigma_{J_n} \varphi_n(I, x') > e_0/4 = \beta > 0.$$

According to the lemma there is a nest of intervals of $[J_n]$. For each interval of the nest $\varphi_n(I_n)/g(I_n) > e_0/8M$. Then $\lim B_n \geq e_0/8M > 0$, contrary to hypothesis. The theorem obviously holds if $\lim$ is substituted for $\lim B$.

Corollary. If $f(I, x)$ is bounded on X and $\lim B f_n(I, x) = f(I, x)$ then $\lim \int_X f_n(I, x) = \int_X f(I, x)$ when the second integral exists.

Proof. Let $\varphi_n(I, x) = |f_n(I, x) - f(I, x)|$. By theorem 4.61 $\varphi_n(I, x)$ is integrable for every n . Also $\Sigma_{\sigma} \varphi_n(I, x) \leq \Sigma_{\sigma} |f_n(I, x)| + \Sigma_{\sigma} |f(I, x)|$. Therefore the sequence $\varphi_n(I, x)$ is essentially uniformly bounded on X . By Theorem 5.21 $\lim \int_X \varphi_n(I, x) = 0$. But $|\int_X f_n(I, x) - \int_X f(I, x)| \leq \int_X \varphi_n(I, x)$.

Theorem 5.22. If $\lim f_n(I, x) = f(I, x)$, where the function on the right is essentially bounded on X , then $\lim \int_X f_n(I, x)$ exists and is unique.

Proof. Let $f'_n(I, x)$ and $f''_n(I, x)$ be any two sub-sequences. Then $\varphi_n(I, x) = |f'_n(I, x) - f''_n(I, x)|$ is a uniformly bounded sequence of integrable functions. If A_n , A'_n , and A''_n are defined by the three equations

$$A_n = \varphi_n(I_n, x) / g(I_n)$$

$$A'_n = |f'_n(I_n, x) - f(I_n, x)| / g(I_n)$$

$$A''_n = |f''_n(I_n, x) - f(I_n, x)| / g(I_n)$$

it is always true that $A_n \leq A'_n + A''_n$. Therefore $\lim \int_X \varphi_n(I, x) = 0$ because $\lim A_n = 0$. Now if $f'_n(I, x)$ and $f''_n(I, x)$ are any two sequences whose limit is $f(I, x)$ let $g(I)$ be equal to the sum of $g'(I)$ and $g''(I)$, the pair of functions relative to which the two sequences converge. Then $\lim \int_X f'_n(I, x) = \lim \int_X f''_n(I, x)$ by the same reasoning. This theorem is not true if $\lim B$ is substituted for $\lim$ in the hypothesis.

Theorem 5.23. If $\lim f_n(I, x) = f(I, x)$ almost everywhere and the sequence $\varphi_n(I, x) = |f_n(I, x) - f(I, x)|$ is equi-absolutely continuous then $\lim f_n(I, x) = f(I, x)$.

Proof. Let E be the set of measure zero associated with the nests on which $\lim A_n \neq 0$ with a given $g(I)$. It is required to find a $g'(I) \geq g(I)$ such that $\lim \varphi_n(I_n, y) / g'(I_n) = 0$ for all nests whose associated points y lie on E . Let $M_1 \supseteq M_2 \dots \supseteq E$ be a sequence of sets of open intervals such that for every K the measure of M_K is less than N_K , a number assured by the equi-uniform continuity of $\varphi_n(I, x)$, for which $\sum_J \varphi_n(I, x) < 1/2^K$ whenever $\sum_J N(I) \leq N_K$. If I is contained in M_K but not in M_{K+1} let $g_0(I) = K(\text{LUB } \sum_O \varphi_n(I, x))$ for all n and subdivisions of I . If I is not contained in M_1 let $g_0(I) = 0$. Then $g'(I) = g(I) + g_0(I)$ is bounded on X because $\sum_O g'(I) = \sum_O g(I) + \sum_O g_0(I)$ is not less than $\sum_O g(I) + \sum_K k/2^K$. The latter series is seen to converge by the ratio test. For every nest H and every K there exists an n_{HK} such that I_n of H is contained in M_K whenever $n \geq n_{HK}$, provided the associated point belongs to E . Then $\varphi_n(I_n, y) / g'(I_n) \leq \varphi_n(I_n, y) / g_0(I_n) \leq 1/K$. Therefore $\lim f_n(I, x) = f(I, x)$.

CHAPTER VI

EXTENDED INTEGRALS

6.1) The A Integral.

Let A be the class of all functions $f(I, x)$ which are essentially bounded and for which there exists a sequence $f_n(I, x)$ such that $\lim f_n(I, x) = f(I, x)$. If $f(I, x)$ is of class A its integral $A \int_X f(I, x)$ is the limit of the integrals of the sequence. This exists and is unique for all such sequences by Theorem 5.22. In consequence of the corollary to Theorem 5.21 the A integral of a function is equal to its σ integral whenever the latter exists.

As a direct result of the properties of limits Theorems 3.38, 3.39, 3.310, 4.51, and 4.61, with their corollaries, are true of the A integral. In the case of Theorem 4.61 it must be shown that the positive and negative parts of the functions of the sequence have separate limits. Denoting these by $P_n(I, x)$, $Q_n(I, x)$; and the corresponding parts of $f(I, x)$ by $P(I, x)$ and $Q(I, x)$, we see that $|P_n(I_n, y) - P(I_n, y)| \leq |f_n(I_n, y) - f(I_n, y)|$; with a similar inequality in Q .

That class A furnishes an extension of the Stieltjes integral may be seen from the following example. Let $f(I, x) = \varphi(x)g(I) = \varphi(x)\Delta g(x)$. Now take $\varphi(x) = 0$, $x \neq x_0$; $\varphi(x_0) = 1$, and $g(x) = 0$, $x \leq x_0$; $g(x) = 1$, $x > x_0$. The σ integral of $f(I, x)$ does not exist because both $\varphi(x)$ and $g(x)$ have a discontinuity on the right at x_0 . Now let $f_n(I, x) = \varphi_n(x)g(I)$, where $\varphi_n(x) = 1$, $x_0 - 1/n \leq x \leq x_0 + 1/n$, and $\varphi_n(x) = 0$ elsewhere. $\lim A_n = \lim |f_n(I_n, y) - f(I_n, y)|/g(I_n) = \lim |\varphi_n(y) - \varphi(y)| = 0$ on every nest. But on every X which contains x_0 as an inner point it is seen that $\sigma \int_X f_n(I, x) = 1$. Therefore $A \int_X f(I, x) = 1$.

An important sub-class of A consists of the absolutely continuous functions of that class. Denote it by A_c . The defining sequence may be taken to be equi-absolutely continuous for any such function. If $f_n(I, x)$ is the given sequence the logical product of $f_n(I, x)$ and $V(f, I)$ for every n furnishes a sequence with the required properties. Every element is integrable σ by

Corollary 3 to Theorem 4.61. It is assumed that $f(I, x)$ is non-negative. We see by Corollary 2 to the same theorem that this assumption may be made without loss of generality in treating functions of class A.

Theorem 6.11. If $f(I, x)$ is of class A_C then $A \int_I f(I, x)$ is absolutely continuous.

Proof. By Theorem 4.71 the sequence of integrals of definition may be taken to be equi-absolutely continuous.

Theorem 6.12. If $f'(I, x) = f''(I, x)$ almost everywhere and both are of class A_C their integrals are equal.

Proof. By Theorem 5.22 the sequence of definition of $f'(I, x)$ also serves to define $f''(I, x)$.

6.2) The C Integral.

The example in the preceding paragraph is an instance of a function equal to zero almost everywhere, and yet its integral is unity on any interval which contains x_0 as an inner point. In order, then, to define an integral which will be independent of the values of the function on sets of zero measure, and to leave open the possibility of extension of integration to unbounded functions, it is evidently necessary to sacrifice its equality with the A integral.

Let C be the class of all functions $f(I, x)$ which have corresponding functions of class A_C to which they are equal almost everywhere. Then the C integral of $f(I, x)$ will be defined as the A integral of the corresponding function. Unless otherwise stated all the integrals henceforth will be understood to be of this type.

Theorem 6.21. If $f'(I, x) = f''(I, x)$ almost everywhere on the measurable set E then $\int_E f'(I, x) = \int_E f''(I, x)$.

Proof. Let us first consider the σ integral. The function $f(I, x) = |f'(I, x) - f''(I, x)|$ is equal to zero on E. For every positive ϵ there exists an m_ϵ such that $\sum_J f(I, x) \leq \epsilon/3$ whenever $m(J) \leq m_\epsilon$, J being any fundamental set. There is also a denumerable set of non-overlapping open intervals, O_ϵ , which contains E, such that $m(O_\epsilon) - m(E) \leq m_\epsilon$. Choose a fundamental set contained in O_ϵ for which $\int_{O_\epsilon} f(I, x) \leq \int_J f(I, x) + \epsilon/3$. Let

n be the number of intervals of this set. If I is one of these intervals there exists a subdivision of it such that $\int_I f(I, x) \leq \Sigma_{\sigma} f(I, x) + e/3n$. Taking J' to be the set of all such intervals of the subdivisions of the elements of J as contain at least one point of E , and J'' the intervals which contain no points of E , it follows that $\int_J f(I, x) \leq \Sigma_J \int_{J'} f(I, x) \leq \Sigma_{J'} f(I, x) + \Sigma_{J''} f(I, x) + e/3$. The first summation is zero when the points of E are inserted. Since $m(J'') \leq m_e$ it has been established that $\int_E f(I, x) \leq \int_{\sigma E} f(I, x) \leq \int_J f(I, x) + e/3 \leq \Sigma_{J'} f(I, x) + 2e/3 \leq e$. This shows that the σ integral on the measurable set E is independent of the values of the function on the complementary set. A glance at the proof of Theorem 5.21 shows that convergence on E is sufficient for the convergence of the integrals of a sequence on E . Therefore the definition of the C integral is consistent and its independence of the values of the function on $C(E)$ follows at once from the same property of the sequence of definition.

Theorems 3.38, 3.39, 3.310, 4.51, 4.61, 4.71, and 4.72 have now been proven explicitly or implicitly for the C integral. Bounded measurable set is to be substituted for interval, and $m(E)$ for $N(I)$, in the statement of the theorems. The property of additivity obviously holds on mutually exclusive sets only.

Theorem 6.22. If $f_n(I, x)$ is an equi-absolutely continuous sequence on the bounded measurable set E then a necessary and sufficient condition that $\lim \int_E f_n(I, x) = 0$ is that there exist a $g(I)$ such that for every positive e the limit of the measure of $E_n[f_n(I, x) > e g(I)]$ be zero.

Necessity. Let E_n be the subset of E such that if y is a point of E_n and e is a positive number then $f_n(I, y) > e N(I)$. $\int_E f_n(I, x) \geq \int_{E_n} f_n(I, x) > e \int_{E_n} N(I) = e \cdot m(E_n)$. No matter what e is chosen $\lim m(E_n) = 0$. Therefore there exists a $g(I) = N(I)$ which satisfies the hypotheses.

Sufficiency. $\int_E f_n(I, x) = \int_{C(E_n)} f_n(I, x) + \int_{E_n} f_n(I, x) \leq e \text{ LUB } \Sigma_{\sigma} g(I) + \int_{E_n} f_n(I, x)$. $\Sigma_{\sigma} g(I)$ is taken on all subdivisions of some interval containing E and is uniformly bounded by definition. The term involving this expression may be made arbitrarily small by the proper choice of e .

The last term vanishes as n increases because of the equi-absolute continuity of the sequence of integrals.

Corollary. If $f_n(I, x)$ is an equi-absolutely continuous sequence of functions of class C , and $\lim f_n(I, x) = 0$ independently of I , then $\lim \int_E f_n(I, x) = 0$.

Proof. Suppose $\lim \int_E f_n(I, x) \neq 0$. No matter what $g(I)$ is chosen there exists an $e_0 > 0$ dependent upon it such that $\lim m(E_n) \neq 0$, where E_n is the set of points y on which $f_n(I, y) > e_0 g(I)$ for some interval containing y . There is a positive e_1 and a subsequence of E_n such that $m(E_n) > e_1$ for every n . By the footnote on page 23 there exists a set E_0 of lower measure not less than e_1 common to an infinity of the sets E_n . For every y of E_0 and every n of this infinite sequence $f_n(I, y) > e_0 g(I)$. Therefore if any $g(I)$ is chosen which makes the sequence A_n of page 23 dependent only upon the point y it is impossible to have $\lim A_n = 0$.

Theorem 6.23. If the point function $\varphi_n(x)$ is uniformly bounded almost everywhere on the bounded measurable set E and $L \int_E \varphi(x) dx$ exists then $f(I, x) = \varphi(x)N(I)$ is of class C and $\int_E f(I, x) = L \int_E \varphi(x) dx$.

Proof. Let $\varphi(x) = 0$ on $C(E)$. There exists a uniformly bounded sequence $\varphi_n(x)$ of continuous functions such that $\lim \varphi_n(x) = \varphi(x)$ almost everywhere. If $g(I) = N(I)$ it is seen that $\lim f_n(I, x) = f(I, x)$ almost everywhere, when $f_n(I, x) = \varphi_n(x)N(I)$ on E . Therefore $f(I, x)$ is of class C because the sequence $f_n(I, x)$ is equi-absolutely continuous and composed of integrable functions. $\int_E f(I, x) = \lim \int_E f_n(I, x) = \lim L \int_E \varphi_n(x) dx = L \int_E \varphi(x) dx$.

6.3) Unbounded Functions.

The C integral can readily be extended to apply to certain classes of unbounded functions by the method of de la Vallée Poussin. Let $f_n(I, x) = f(I, x)$ for every x such that $f(I, x) \leq n \cdot N(I)$. For all other points of I let $f_n(I, x) = n \cdot N(I)$. If $\int_E f_n(I, x)$ exists with every n , and if the sequence of integrals has a limit as n increases, then the integral of $f(I, x)$ is defined to be equal to this limit. The function $N(I)$ is introduced arbitrarily in order to have the integral of a

function of the form $\phi(x)\Delta x$ be equal to the corresponding Lebesgue integral. If a non-absolutely continuous function were to be used in place of $N(I)$ a different integral would be obtained whose value would depend not only upon the integrand but upon this comparison function as well.

6.4) Relative Absolute Continuity.

The measure of certain point sets may be defined relative to an additive interval function $g(I)$ of the type defined in paragraph 5.1). Let $g(O)$ be the LUB of $\sum_J g(I)$ on all fundamental sets J contained in the open set O . Then $\bar{g}(E)$, the exterior measure of the bounded point set E , will be the GLB of $g(O)$ for all open sets O which contain E . The set E is measurable relative to $g(I)$ if the exterior measure of the complementary set with respect to X is equal to $g(X) - \bar{g}(E)$, where X is an interval containing E .

The function $f(I, x)$ will be said to be absolutely continuous relative to the additive function $g(I)$ if $\lim \sum_J f(I, x) = 0$ on all fundamental sets J as $\sum_J g(I)$ goes to zero. In a theory of integration which is concerned only with bounded functions no distinction need be made between absolute continuity relative to $N(I)$ and that relative to any other additive function $g(I)$. In the theorems and definitions of the foregoing paragraphs which deal with absolute continuity and measure it is permissible to insert the words, "relative to the additive function $g(I)$ ". It must be noticed that relative to some such functions a closed interval may not be measurable. Since $f(I, x)$ is absolutely continuous relative to $\int_I |f(I, x)|$, an additive function, a convenient comparative function is always available.

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